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Lower bounds for the eigenvalues of the Dirac operator ★

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Abstract

On a compact Riemannian spin manifold we give new lower bounds for the eigenvalues of the Dirac operator in terms of the curvature and of the norm of an appropriate endomorphism of the tangent bundle. As a corollary, one gets all known results in this direction. The limiting-case is then studied.

Keywords: Dirac operator; Eigenvalues;
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1. Introduction

T. Friedrich [Fr 1] proved with the help of the Lichnerowicz formula [Li 1] that, on a compact Riemannian spin manifold (M^n, g) of dimension $n \geq 2$, any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S, \qquad (*)$$

where S is the scalar curvature of (M^n, g). In 1984 the author [Hi 1] improved $(*)$ by showing that, for $n \geq 3$

$$\lambda^2 \geq \frac{n}{4(n-1)} \mu_1, \qquad (**)$$

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where μ_1 is the first eigenvalue of the conformal Laplacian acting on functions. Inequalities $(*)$ and $(**)$ contain information only in the case where the scalar curvature is positive. In this paper we prove the following:

Theorem A. *On a compact Riemannian spin manifold (M^n, g) of dimension $n \geq 2$, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geq \inf_M \left(\tfrac{1}{4} S + |\ell_\psi|^2\right),$$

where S is the scalar curvature of (M^n, g), and ℓ_ψ is the field of symmetric endomorphisms of the tangent bundle associated with the field of quadratic forms defined on the complement of the set of zeroes of ψ, for any vector field X, by

$$Q_\psi(X) = \mathrm{Re}\,(X \cdot \nabla_X \psi, \psi/|\psi|^2)\,.$$

Theorem B. *On a compact Riemannian spin manifold (M^n, g) of dimension $n \geq 3$, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geq \tfrac{1}{4}\mu_1 + \inf_M |\ell_\psi|^2,$$

where μ_1 is the first eigenvalue of the Yamabe operator, i.e.,

$$L := 4\frac{n-1}{n-2}\Delta + S$$

acting on functions.

Corollary C. *Under the same conditions as in Theorem A, one has*

$$\lambda^2 \geq \tfrac{1}{4}\mu\,\{\mathrm{Vol}(M, g)\}^{-2/n} + \inf_M |\ell_\psi|^2,$$

where μ is a conformal invariant, called the Yamabe number, defined by

$$\mu := \inf_{n\geq 2}\ \inf_{\overline{g}\in[g]} \frac{\int_M \overline{S} v_{\overline{g}}}{\{\mathrm{Vol}(M, \overline{g})\}^{(n-2)/n}} := \inf_{n\geq 3}\ \inf_{h>0} \frac{\int_M h L(h) v_g}{\|h\|^2_{2n/(n-2)}}\,.$$

Corollary D. *For $n = 2$, under the same conditions as in Theorem A, one has*

$$\lambda^2 \geq \frac{2\pi\chi(M)}{\mathrm{Area}(M)} + \inf_M |\ell_\psi|^2,$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M.

Proposition E. *On a compact Riemannian spin manifold (M^n, g) of dimension $n \geq 3$, assume that an eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 = \tfrac{1}{4}\mu_1 + \inf_M |\ell_\psi|^2\,.$$

Then, $|\ell_\psi|$ is constant, and if u_1 denotes an eigenfunction of the Yamabe operator corresponding to μ_1, then for any vector field X

$$g(X, \ell_\psi(du_1) - \lambda du_1) = g(\lambda X - \ell_\psi(X), du_1) = 0.$$

2. Notations and preliminaries

Let (M^n, g) be an n-dimensional Riemannian spin manifold. The Levi-Civita connection on the tangent bundle TM is denoted by ∇. The same symbol is used to denote its natural extension on the bundle of exterior forms, on the bundle of endomorphisms of the tangent bundle, and on the complex spin bundle ΣM. The spin bundle carries a natural Hermitian scalar product denoted by $(\ , \ )$. The linear isomorphism between the algebra of exterior forms and the Clifford algebra allows to consider the action of exterior forms on the spin bundle via Clifford multiplication. For any vector field X, any k-form α, and any spinor fields ψ and φ, one has (see [LM])

$$X(\psi, \varphi) = (\nabla_X \psi, \varphi) + (\psi, \nabla_X \varphi), \tag{1}$$

$$\nabla_X(\alpha \cdot \psi) = (\nabla_X \alpha) \cdot \psi + \alpha \cdot \nabla_X \psi, \tag{2}$$

$$(\alpha \cdot \psi, \varphi) = (-1)^{k(k+1)/2} (\psi, \alpha \cdot \varphi), \tag{3}$$

$$X \cdot \alpha = X \wedge \alpha - \iota_X \alpha, \tag{4}$$

where "$\wedge$" is the exterior product and "ι_X" is the interior product with X. The Dirac operator D acting on sections of the spin bundle is locally defined by

$$D = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i},$$

with $\{e_1, \dots, e_n\}$ a local orthonormal basis of the tangent bundle TM. At any point x in M, we choose normal coordinates at this point so that $(\nabla e_i)(x) = 0$, for all $i \in \{1, \dots, n\}$. All the computations will be made in such charts. The Dirac operator satisfies the Lichnerowicz formula

$$\nabla^* \nabla = D^2 - \tfrac{1}{4} S, \tag{5}$$

where ∇^* stands for the adjoint of ∇ with respect to the global scalar product.

3. Modification of the Levi-Civita connection

Definition 1. Let ℓ be a symmetric endomorphism of the tangent bundle. For any tangent vector field X and any spinorfield ψ, define the modified connection ∇^ℓ by

$$\nabla^\ell_X \psi := \nabla_X \psi + \ell(X) \cdot \psi. \tag{6}$$

One has the following

Lemma 1. *The modified connection ∇^ℓ is a metric connection, i.e., for any tangent vector field X, and any spinor fields ψ and φ, one has*

$$X(\psi,\varphi) = (\nabla^\ell_X\psi,\varphi) + (\psi,\nabla^\ell_X\varphi).$$

Proof. With the identification of 1-forms with vector fields via the Riemannian metric, it is sufficient to use Eq. (2) and Eq. (3) with $k = 1$. □

Lemma 2. *The operator $\nabla^{\ell^*}\nabla^\ell : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is non-negative and formally self-adjoint. In particular, on a compact manifold one has*

$$\int_M (\nabla^{\ell^*}\nabla^\ell\psi,\varphi) = \int_M (\nabla^\ell\psi,\nabla^\ell\varphi),$$

for all sections of ΣM.

Proof. By definition, it follows that

$$\begin{aligned}(\nabla^{\ell^*}\nabla^\ell\psi,\varphi) &= -\sum_{i=1}^n(\nabla^\ell_{e_i}\nabla^\ell_{e_i}\psi,\varphi)\\ &= -\sum_{i=1}^n\{e_i(\nabla^\ell_{e_i}\psi,\varphi) - (\nabla^\ell_{e_i}\psi,\nabla^\ell_{e_i}\varphi)\}\\ &= \mathrm{div}(X) + (\nabla^\ell_{e_i}\psi,\nabla^\ell_{e_i}\varphi),\end{aligned}$$

where X is the tangent vector field defined by the following condition:

$$\forall Y \in \Gamma(TM), \qquad g(X,Y) = (\nabla^\ell_Y\psi,\varphi).$$

In fact, at a given point, with respect to normal coordinates, we get for the divergence term

$$\begin{aligned}\mathrm{div}(X) &= -\sum_{i=1}^n g(\nabla_{e_i}X,e_i)\\ &= -\sum_{i=1}^n e_i\, g(X,e_i)\\ &= -\sum_{i=1}^n e_i(\nabla^\ell_{e_i}\psi,\varphi).\end{aligned}$$

This yields Lemma 2 by integration. □

Proposition 3. *For any linear endomorphism ℓ of the tangent bundle, and for any spinor field ψ, the following identity holds:*

$$|\nabla^\ell\psi|^2 := \sum_{i=1}^n(\nabla^\ell_{e_i}\psi,\nabla^\ell_{e_i}\psi)$$

$$= |\nabla\psi|^2 + |\ell|^2\,|\psi|^2 - 2\,\mathrm{Re}\sum_{i=1}^{n}(\,\ell(e_i)\cdot\nabla_{e_i}\psi,\psi)\,. \tag{7}$$

Proof. Using (3) and (5) with $k = 1$, it follows that

$$\begin{aligned}\sum_{i=1}^{n}|\nabla^{\ell}_{e_i}\psi|^2 &= \sum_{i=1}^{n}(\nabla_{e_i}\psi + \ell(e_i)\cdot\psi, \nabla_{e_i}\psi + \ell(e_i)\cdot\psi)\\ &= |\nabla\psi|^2 + \sum_{i=1}^{n}(\nabla_{e_i}\psi, \ell(e_i)\cdot\psi) + \sum_{i=1}^{n}(\ell(e_i)\cdot\psi, \nabla_{e_i}\psi)\\ &\quad + \sum_{i=1}^{n}(\ell(e_i)\cdot\psi, \ell(e_i)\cdot\psi)\\ &= |\nabla\psi|^2 - 2\,\mathrm{Re}\sum_{i=1}^{n}(\,\ell(e_i)\cdot\nabla_{e_i}\psi,\psi) + |\ell|^2\,|\psi|^2.\end{aligned}$$

□

4. Eigenvalues estimate

We now show that for an appropriate choice of the symmetric endomorphism ℓ in Proposition 3, one gets a sharp estimate of the first eigenvalue of the Dirac operator on compact Riemannian manifolds. For this we need the following:

Definition 4. On the complement of the set of zeroes of a spinorfield ψ, define for any tangent vector fields X and Y, the symmetric bilinear tensor Q_ψ by

$$Q_\psi(X,Y) = \tfrac{1}{2}\,\mathrm{Re}\,(X\cdot\nabla_Y\psi + Y\cdot\nabla_X\psi, \psi/|\psi|^2). \tag{8}$$

We choose the local orthonormal frame $\{e_1,\ldots,e_n\}$ so that Q_ψ is diagonal. For the associated field of quadratic forms, it follows that

$$Q_\psi(e_i) = \mathrm{Re}\,(e_i\cdot\nabla_{e_i}\psi, \psi/|\psi|^2),$$

and $\mathrm{tr}\,Q_\psi = \mathrm{Re}(D\psi, \psi/|\psi|^2)$. If ψ is such that $D\psi = \lambda\psi$, then $\mathrm{tr}\,Q_\psi = \lambda$.

Theorem 5. *On a compact Riemannian spin manifold (M^n, g) of dimension $n \geq 2$, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geq \inf_M(\tfrac{1}{4}S + |\ell_\psi|^2), \tag{9}$$

where S is the scalar curvature of (M^n, g), and ℓ_ψ is the field of symmetric endomorphisms associated with the field of quadratic forms

$$Q_\psi(e_i) = \mathrm{Re}\,(e_i\cdot\nabla_{e_i}\psi, \psi/|\psi|^2)\,.$$

Before giving the proof of Theorem 5, we give the following immediate lemma:

Lemma 6. *On the space of fields of quadratic forms, $\Gamma(T^\star M \otimes T^\star M)$, consider the function f defined by*

$$f(q) = -2\,g(q, Q_\psi) + g(q,q)\,,$$

where g is the natural extension of the Riemannian metric to $\Gamma(T^\star M \otimes T^\star M)$. Then, f attains its minimum at $q = Q_\psi$.

Proof of Theorem 5. Integration of Eq. (7) for $\ell = \ell_\psi$, together with Lemma 6, gives Theorem 6. □

Before investigating the limiting case of (9), we compare inequality (9) with previous results in this direction.

Corollary 7 ([Fr 1]). *Under the same conditions as in Theorem 5, one has*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S\,. \tag{10}$$

Proof. It is sufficient to notice, by the Cauchy–Schwarz inequality, that

$$|\ell_\psi|^2 \geq \frac{(\operatorname{tr}\ell_\psi)^2}{n} = \frac{\lambda^2}{n}.$$ □

Remark 8. Inequality (10) is of interest only in the case where $\inf_M S$ is positive, while (9) may still give information on the first eigenvalue in the case where $\inf_M S$ is non-positive.

5. Conformal geometry and eigenvalues estimate

In this section, we apply the techniques in [Hi 2] and [Hi 3] concerning conformal changes of the Riemannian metric to get a sharper estimate than (9) in terms of the first eigenvalue of the Yamabe operator. We use the notations and the results in [Hi 2].

The main result of this section is the following:

Theorem 9. *On a compact Riemannian spin manifold (M^n, g) of dimension $n \geq 3$, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geq \tfrac{1}{4}\,\mu_1 + \inf_M |\ell_\psi|^2, \tag{11}$$

where μ_1 is the first eigenvalue of the Yamabe operator, i.e.,

$$L := 4\frac{n-1}{n-2}\,\Delta + S$$

acting on functions, and ℓ_ψ is the field of symmetric endomorphisms associated with the field of quadratic forms

$$Q_\psi(X) = \operatorname{Re}\left(X \cdot \nabla_X \psi, \psi/|\psi|^2\right).$$

Using the Cauchy–Schwarz inequality it follows that

Corollary 10 ([Hi 1]). *Under the same conditions as in Theorem 9, any eigenvalue of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1 \,. \tag{12}$$

The conformal class of g is *denoted* by $[g]$. An element in this class is a metric $\overline{g}$ such that $\overline{g} = e^{2u}g$, for a real function u on M. At a given point x of M, we consider a g-orthonormal basis $e = \{e_1, \dots, e_n\}$ of T_xM. The corresponding $\overline{g}$-orthonormal basis is *denoted* by $\overline{e} = \{e^{-u}e_1, \dots, e^{-u}e_n\}$. This correspondence extends to the spin level to give an isometry between the corresponding spinor bundles. We agree to put a $(\overline{})$ above every object which is naturally associated with the metric $\overline{g}$. Then, for any spinorfields ψ and φ, one has

$$(\overline{\psi}, \overline{\varphi}) = (\psi, \varphi)\,,$$

where (,) *denotes* the natural Hermitian scalar products on $\Gamma(\Sigma M)$, and on $\Gamma(\Sigma \overline{M})$ which make an isometry from the isomorphism between these spaces.The Levi-Civita connection $\overline{\nabla}$ is related to ∇ by

$$\overline{\nabla}_X \overline{\psi} = \overline{\nabla_X\psi} - \tfrac{1}{2}\,\overline{X \cdot du \cdot \psi} - \tfrac{1}{2}\,X(u)\overline{\psi}\,. \tag{13}$$

According to [Hn, Ba] the Dirac operators satisfy

$$\overline{D}\,(\,e^{-(n-1)u/2}\,\overline{\psi}\,) = e^{-(n+1)u/2}\,\overline{D\psi}\,. \tag{14}$$

For $D\psi = \lambda\psi$, one gets $\overline{D}\,\overline{\varphi} = \lambda e^{-u}\,\overline{\varphi}$, with $\varphi = e^{-(n-1)u/2}\,\psi$. We *denote* by $|\ell_{\overline{\varphi}}|$ the norm of the field of endomorphisms $\ell_{\overline{\varphi}}$ w.r.t. the metric defined by $\overline{g}$, while $|\ell_\varphi|$ is the norm computed w.r.t. the metric g.

Proposition 11. *The following relations hold:*

$$|\ell_{\overline{\varphi}}|^2 = e^{-2u}\,|\ell_\varphi|^2 = e^{-2u}\,|\ell_\psi|^2, \tag{15}$$

where $\varphi = e^{-(n-1)u/2}\,\psi$, *and* $\overline{g} = e^{2u}g$.

Proof. With the help of identity (13) and the definition of $Q_{\overline{\varphi}}(\overline{e}_i, \overline{e}_j)$, one gets

$$\begin{aligned} Q_{\overline{\varphi}}(\overline{e}_i, \overline{e}_j) &= \tfrac{1}{2}\,\mathrm{Re}\,(\overline{e}_i \,\bar{\cdot}\, \overline{\nabla}_{\overline{e}_j}\overline{\varphi} + \overline{e}_j \,\bar{\cdot}\, \overline{\nabla}_{\overline{e}_i}\overline{\varphi}, \overline{\varphi}/|\overline{\varphi}|^2) \\ &= \tfrac{1}{2}\,e^{-u}\,\mathrm{Re}\,(\overline{e}_i \,\bar{\cdot}\, \overline{\nabla}_{e_j}\overline{\varphi} + \overline{e}_j \,\bar{\cdot}\, \overline{\nabla}_{e_i}\overline{\varphi}, \overline{\varphi}/|\overline{\varphi}|^2) \\ &= \tfrac{1}{2}\,e^{-u}\,\mathrm{Re}\,(\overline{e}_i \,\bar{\cdot}\, [\,\overline{\nabla_{e_j}\varphi} - \tfrac{1}{2}\,\overline{e_j \cdot du \cdot \varphi} - \tfrac{1}{2}\,e_j(u)\,\overline{\varphi}\,] \\ &\qquad + \overline{e}_j \,\bar{\cdot}\, [\,\overline{\nabla_{e_i}\varphi} - \tfrac{1}{2}\,\overline{e_i \cdot du \cdot \varphi} - \tfrac{1}{2}\,e_i(u)\,\overline{\varphi}\,], \overline{\varphi}/|\overline{\varphi}|^2)\,, \end{aligned} \tag{16}$$

which, after using $e_i \cdot e_j + e_j \cdot e_i = -2\,\delta_{ij}$ and (3), gives with $k = 1$

$$\begin{aligned} Q_{\overline{\varphi}}(\overline{e}_i, \overline{e}_j) &= \tfrac{1}{2}\,e^{-u}\,\mathrm{Re}\,(\overline{e}_i \,\bar{\cdot}\, \overline{\nabla_{e_j}\varphi} + \overline{e}_j \,\bar{\cdot}\, \overline{\nabla_{e_i}\varphi}, \overline{\varphi}/|\overline{\varphi}|^2) \\ &= \tfrac{1}{2}\,e^{-u}\,\mathrm{Re}\,(\overline{e_i \cdot \nabla_{e_j}\varphi + e_j \cdot \nabla_{e_i}\varphi}, \overline{\varphi}/|\overline{\varphi}|^2)\,. \end{aligned}$$

Since the isomorphism between the spin bundles associated with the metrics g and $\overline{g}$ is an isometry w.r.t. the associated Hermitian scalar product, it follows that

$$\begin{aligned} Q_{\overline{\varphi}}(\overline{e}_i,\overline{e}_j) &= \tfrac{1}{2}\, e^{-u}\, \mathrm{Re}\,(e_i\cdot\nabla_{e_j}\varphi + e_j\cdot\nabla_{e_i}\varphi, \varphi/|\varphi|^2) \\ &= e^{-u}\, Q_\varphi(e_i,e_j)\,. \end{aligned} \tag{17}$$

On the other hand, since $\varphi = e^{-(n-1)u/2}\,\psi$, identity (3) implies

$$\begin{aligned} Q_\varphi(e_i,e_j) &= \tfrac{1}{2}\,\mathrm{Re}\,(e_i\cdot\nabla_{e_j}\varphi + e_j\cdot\nabla_{e_i}\varphi, \varphi/|\varphi|^2) \\ &= \tfrac{1}{2}\, e^{-(n-1)u}\,\mathrm{Re}\,(e_i\cdot\nabla_{e_j}\psi + e_j\cdot\nabla_{e_i}\psi, \psi/|\varphi|^2) \\ &\qquad - \tfrac{1}{4}(n-1)\, e^{-(n-1)u}\,\mathrm{Re}\,(e_j(u)\, e_i\cdot\psi + e_i(u)\, e_j\cdot\psi, \psi/|\varphi|^2) \\ &= \tfrac{1}{2}\,\mathrm{Re}\,(e_i\cdot\nabla_{e_j}\psi + e_j\cdot\nabla_{e_i}\psi, \psi/|\psi|^2) \\ &= Q_\psi(e_i,e_j)\,, \end{aligned}$$

hence, with (17), one gets

$$Q_{\overline{\varphi}}(\overline{e}_i,\overline{e}_j) = e^{-u}\, Q_\psi(e_i,e_j)\,. \tag{18}$$

To finish the proof of Proposition 11, we compare $\ell_{\overline{\varphi}}$ with ℓ_ψ. By (18) and the relation between ℓ_ψ and Q_ψ, one gets

$$\begin{aligned} \overline{g}(\ell_{\overline{\varphi}}(\overline{e}_i),\overline{e}_j) &= Q_{\overline{\varphi}}(\overline{e}_i,\overline{e}_j) \\ &= e^{-u} Q_\psi(e_i,e_j) \\ &= e^{-u} g(\ell_\psi(e_i),e_j)\,. \end{aligned}$$

The above identity implies

$$\ell_{\overline{\varphi}} = e^{-u}\,\ell_\psi\,,$$

which gives (15). □

Proof of Theorem 9. Proposition 3 with respect to the metric $\overline{g} = e^{2u}g$, applied to the spinorfield $\overline{\varphi}$ and for $\ell = \ell_{\overline{\varphi}}$ gives after integration

$$\int_M |\overline{\nabla}^{\ell_{\overline{\varphi}}}\,\overline{\varphi}|^2 = \int_M |\overline{D}\,\overline{\varphi}|^2 - \int_M (\tfrac{1}{4}\overline{S} + |\ell_{\overline{\varphi}}|^2)|\overline{\varphi}|^2, \tag{19}$$

where $\overline{S}$ is the scalar curvature of $(M^n,\overline{g})$. For $\varphi = e^{-(n-1)u/2}\,\psi$ and $D\psi = \lambda\psi$, one gets $\overline{D}\,\overline{\varphi} = \lambda e^{-u}\,\overline{\varphi}$, which when combined with Proposition 11, implies

$$\begin{aligned} \int_M |\overline{\nabla}^{\ell_{\overline{\varphi}}}\,\overline{\varphi}|^2 &= \int_M [\lambda^2 - (\tfrac{1}{4}\overline{S}\, e^{2u} + |\ell_{\overline{\varphi}}|^2\, e^{2u})]\, e^{-2u}|\overline{\varphi}|^2 \\ &= \int_M [\lambda^2 - (\tfrac{1}{4}\overline{S}\, e^{2u} + |\ell_\psi|^2)]\, e^{-2u}\,|\overline{\varphi}|^2. \end{aligned} \tag{20}$$

For any function u, Eq. (20) implies

$$\lambda^2 \geq \inf_M (\tfrac{1}{4}\overline{S}\, e^{2u} + |\ell_\psi|^2)$$

$$\geq \tfrac{1}{4} \inf_M (\overline{S} e^{2u}) + \inf_M |\ell_\psi|^2$$
$$\geq \tfrac{1}{4} \sup_u \inf_M (\overline{S} e^{2u}) + \inf_M |\ell_\psi|^2.$$

The relation between S and $\overline{S}$ for $\overline{g} = e^{2u} g = h^{4/(n-2)} g$ is given by

$$\begin{aligned} h^{-1} L(h) &:= 4 \frac{n-1}{n-2} h^{-1} \Delta h + S \\ &= \overline{S} h^4/(n-2) = \overline{S} e^{2u} = 2(n-1) \Delta u - (n-1)(n-2)\, |du|^2 + S. \end{aligned}$$

For $u = u_1$ the function corresponding to an eigenfunction h_1 of L associated with its first eigenvalue μ_1, it follows that $\mu_1 = \overline{S} e^{2u_1}$. In fact, one has

$$\mu_1 = \sup_u \inf_M (\overline{S} e^{2u}) ,$$

and the supremum is achieved by a function u, if and only if u is an eigenfunction of the Yamabe operator associated with μ_1 (see [Hi 3]). □

Corollary 12. *Under the same conditions as in Theorem 9, one has*

$$\lambda^2 \geq \tfrac{1}{4} \mu \{\mathrm{Vol}(M, g)\}^{-2/n} + \inf_M |\ell_\psi|^2, \tag{21}$$

where μ is a conformal invariant, called the Yamabe number, defined by

$$\mu := \inf_{n \geq 2} \inf_{\overline{g} \in [g]} \frac{\int_M \overline{S} v_{\overline{g}}}{\{\mathrm{Vol}(M, \overline{g})\}^{(n-2)/n}} := \inf_{n \geq 3} \inf_{h > 0} \frac{\int_M h L(h) v_g}{\|h\|^2_{2n/(n-2)}}. \tag{22}$$

Proof. It is sufficient to use the Hölder inequality to show that

$$\mu_1 \geq \mu \{\mathrm{Vol}(M, g)\}^{-2/n}, \tag{23}$$

which, when combined with (11), yields (21). □

Corollary 13. *For $n = 2$, under the same conditions as in Theorem 9, one has*

$$\lambda^2 \geq \frac{2\pi\chi(M)}{\mathrm{Area}(M)} + \inf_M |\ell_\psi|^2 , \tag{24}$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M.

Proof. For $n = 2$, by (22), the Gauss–Bonnet formula, and the variation of the scalar curvature in a conformal class, it follows that

$$\mu = \int_M \overline{S} v_{\overline{g}} = \int_M \overline{S} e^{2u} v_g = \int_M (S + 2 \Delta u)\, v_g = 2\pi\chi(M) ,$$

where v_g is the volume element attached with g (compare with [Bä 1] and [Hi 3]). □

6. Manifolds with small eigenvalues

We start with the following proposition, which will be used in the investigation of the limiting case of (11).

Proposition 14. *For a non-trivial spinor ψ, if $\nabla^{\ell_\psi}\psi \equiv 0$, then $|\psi|^2$ is constant, and*

$$(\operatorname{tr}\ell_\psi)^2 := f^2 = \tfrac{1}{4}S + |\ell_\psi|^2, \tag{25}$$

$$\operatorname{grad} f = -\operatorname{div}\ell_\psi. \tag{26}$$

Proof. Lemma 1 and the condition $\nabla^{\ell_\psi}\psi \equiv 0$ imply that $|\psi|^2$ is constant. The curvature tensor on the spin bundle associated with the connection ∇ (resp. ∇^{ℓ_ψ}) is *denoted* by $\mathbf{R}$ (resp. $\mathbf{R}^{\ell_\psi}$). One easily gets the following relation:

$$\mathbf{R}^{\ell_\psi}_{X,Y}\psi = \mathbf{R}_{X,Y}\psi + d\ell_\psi(X,Y)\cdot\psi + [\ell_\psi(X),\ell_\psi(Y)]\cdot\psi, \tag{27}$$

where $d\ell_\psi$ is a 2-form with values in $\Gamma(TM)$ and $[X,Y] = X\cdot Y - Y\cdot X$. Taking $Y = e_i$ in (27) and performing its Clifford multiplication by e_i yields

$$\begin{aligned}\sum_i e_i\cdot\mathbf{R}^{\ell_\psi}_{X,e_i}\psi &= -\tfrac{1}{2}\operatorname{Ric}(X)\cdot\psi + \sum_i e_i\cdot d\ell_\psi(X,e_i)\cdot\psi \\ &\quad + \sum_i e_i\cdot[\ell_\psi(X),\ell_\psi(e_i)]\cdot\psi,\end{aligned} \tag{28}$$

where Ric *denotes* the Ricci endomorphism on the tangent bundle. We then decompose the last two terms in (28) using (4) in order to separate, when one takes the scalar product of (28) with ψ, real functions from imaginary ones. With the help of (4), it follows that

$$\begin{aligned}&\sum_i e_i\cdot d\ell_\psi(X,e_i)\cdot\psi \\ &\quad= \sum_i e_i\cdot[(\nabla_X\ell_\psi)(e_i) - (\nabla_{e_i}\ell_\psi)(X)]\cdot\psi \\ &\quad= \sum_i[e_i\wedge(\nabla_X\ell_\psi)(e_i) - e_i\wedge(\nabla_{e_i}\ell_\psi)(X)]\cdot\psi \\ &\qquad - \sum_i[(\nabla_X\ell_\psi)(e_i,e_i) - (\nabla_{e_i}\ell_\psi)(X,e_i)]\cdot\psi \\ &\quad= \sum_i[e_i\wedge d\ell_\psi(X,e_i)]\cdot\psi - [X(\operatorname{tr}\ell_\psi) + \operatorname{div}\ell_\psi(X)]\psi.\end{aligned} \tag{29}$$

For the last term, since $X\cdot Y + Y\cdot X = -2g(X,Y)$, one gets

$$\begin{aligned}&\sum_i e_i\cdot[\ell_\psi(X),\ell_\psi(e_i)]\cdot\psi \\ &\quad= \sum_i e_i\cdot[\ell_\psi(X)\cdot\ell_\psi e_i) - \ell_\psi(e_i)\cdot\ell_\psi(X)]\cdot\psi\end{aligned}$$

$$= -\sum_i [\ell_\psi(X)\cdot e_i\cdot \ell_\psi(e_i) + 2g(\ell_\psi(X),e_i)]\cdot\psi$$
$$-\sum_i e_i\cdot\ell_\psi(e_i)\cdot\ell_\psi(X)\cdot\psi$$
$$= 2(\operatorname{tr}\ell_\psi)\,\ell_\psi(X)\cdot\psi - 2\sum_i g(X,\ell_\psi(e_i))\,\ell_\psi(e_i)\cdot\psi\,. \tag{30}$$

Combining Eqs. (28), (29) and (30), one gets

$$\tfrac{1}{2}\operatorname{Ric}(X)\cdot\psi = \sum_i [e_i\wedge d\ell_\psi(X,e_i)]\cdot\psi + 2(\operatorname{tr}\ell_\psi)\,\ell_\psi(X)\cdot\psi$$
$$-2\sum_i g(X,\ell_\psi(e_i))\,\ell_\psi(e_i)\cdot\psi - (X(\operatorname{tr}\ell_\psi) + \operatorname{div}\ell_\psi(X))\psi\,. \tag{31}$$

Taking the scalar curvature of (31) with ψ, and after separating real and imaginary parts, yields for every field X the relations

$$X(\operatorname{tr}\ell_\psi) = -\operatorname{div}\ell_\psi(X),$$
$$\tfrac{1}{2}\operatorname{Ric}(X)\cdot\psi = \sum_i [e_i\wedge d\ell_\psi(X,e_i)]\cdot\psi + 2(\operatorname{tr}\ell_\psi)\,\ell_\psi(X)\cdot\psi$$
$$-2\sum_i g(X,\ell_\psi(e_i))\,\ell_\psi(e_i)\cdot\psi. \tag{33}$$

Identity (32) is equivalent to (26). Clifford multiplication of (33) with e_j, and for $X = e_j$, gives

$$-\tfrac{1}{2}S\psi = \sum_{i,j} e_i\cdot(e_j\wedge d\ell_\psi(e_i,e_j))\cdot\psi - 2(\operatorname{tr}\ell_\psi)^2\psi + 2\|\ell_\psi\|^2\psi\,. \tag{34}$$

Identity (4) implies

$$\sum_{i,j} e_i\cdot(e_j\wedge d\ell_\psi(e_i,e_j))\cdot\psi = \left(\sum_{i,j} e_i\wedge e_j\wedge d\ell_\psi(e_i,e_j)\right)\cdot\psi$$
$$-\sum_{i,j}\iota_{e_i}(e_j\wedge d\ell_\psi(e_i,e_j))\cdot\psi\,. \tag{35}$$

The first term of the r.h.s. of (35) is zero since ℓ_ψ is symmetric, and the last term is the Clifford multiplication of ψ with a vector field, which gives an imaginary function when taking its scalar product with ψ. Hence the scalar product of (34) with ψ is precisely (25). □

We now study the limiting case of (11).

Proposition 15. *On a compact Riemannian spin manifold (M^n, g), assume that an eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 = \tfrac{1}{4}\mu_1 + \inf_M |\ell_\psi|^2\,.$$

Then, $|\ell_\psi|$ is constant, and if u_1 denotes an eigenfunction of the Yamabe operator corresponding to μ_1, then for any vector field X

$$g(X, \ell_\psi(du_1) - \lambda du_1) = g(\lambda X - \ell_\psi(X), du_1) = 0. \tag{35}$$

Proof. If equality in (11) holds, then Eq. (20) with $u = u_1$ gives $|\ell_\psi|$ is constant, and $\overline{\nabla}^{\ell_{\overline{\varphi}}} \overline{\varphi} \equiv 0$. By Proposition 14, we have

$$(\operatorname{tr} \ell_{\overline{\varphi}})^2 := f^2 = \tfrac{1}{4}\overline{S} + |\ell_{\overline{\varphi}}|^2, \tag{36}$$

$$\operatorname{grad} f = -\operatorname{div} \ell_{\overline{\varphi}}. \tag{37}$$

Eq. (36) together with Proposition 11, imply equality in (11). It is straightforward to see that Proposition 11, combined with (37), gives (35). □

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